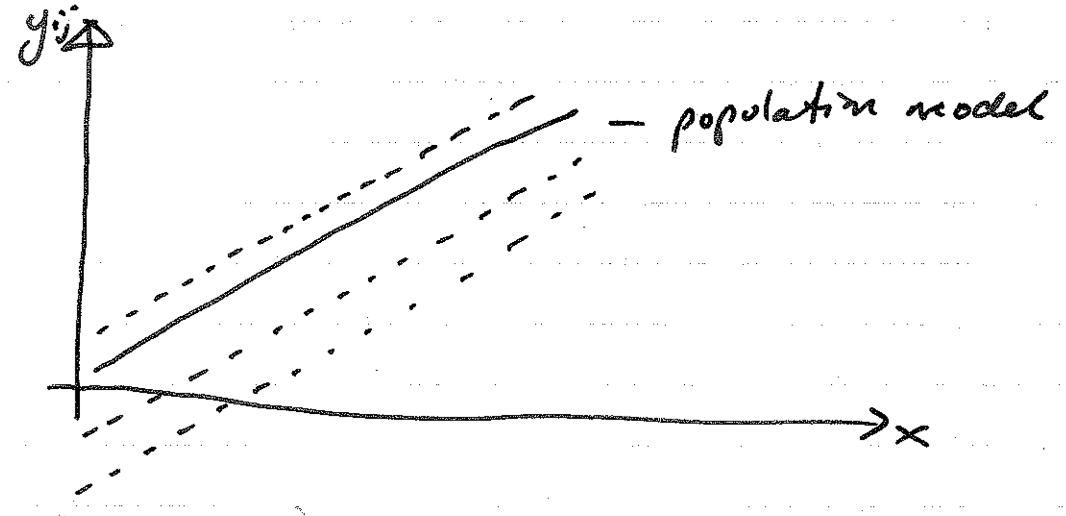
LME MODELS:

1. Random Intercepts:

$$y_{ij} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + u_{0i} + \varepsilon_{ij}$$

$$u_{0i} \sim N(0, \sigma_u^2)$$
$$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$$



2. Random Slopes:

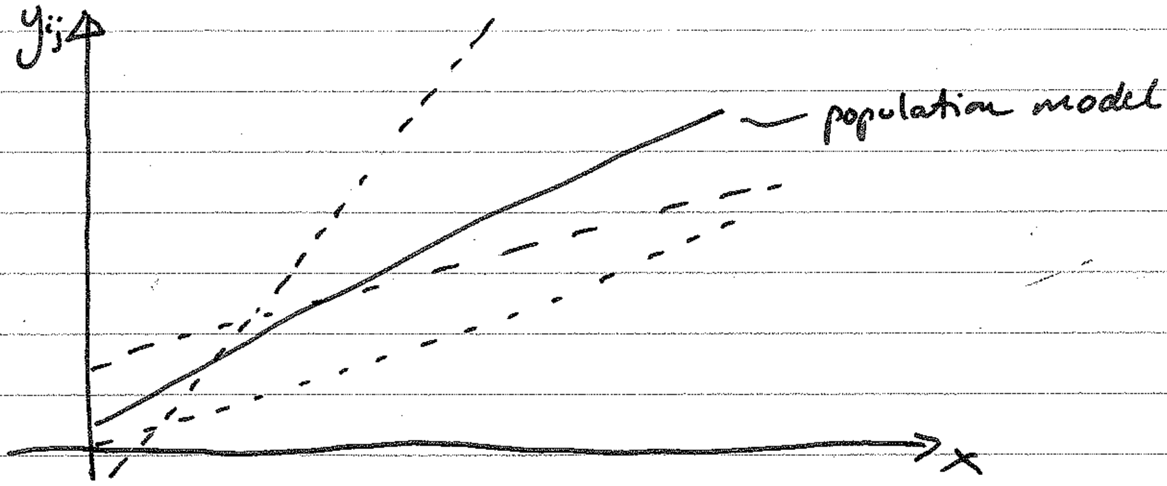
$$y_{ij} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + u_{0i} + u_{1i} X + \varepsilon_{ij}$$

$$\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$$

$$(u_{0i}, u_{1i}) \sim N(\underset{\sim}{0}, \Sigma)$$

where Σ is a variance-covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_{u_0}^2 & \sigma_{01} \\ \sigma_{01} & \sigma_{u_1}^2 \end{bmatrix}$$



BLUPS:

Each group or cluster will deviate from the overall population model (as described by the fixed effects model) by an amount determined by their individual random effects.

The random effects are unknown, but they can be estimated via BLUPS - Best Linear Unbiased Predictors. We call them predictors rather than estimates (as we call the $\hat{\beta}$s) to distinguish them from fixed effects - random effects are random, so we predict them rather than estimate them.

Eg: Suppose the fitted regression model is

$$\hat{y} = 2 + 12.7X.$$

Suppose we had fitted a random intercepts model & individual 5 had a BLUP of -1. This means their individual curve is

$$\hat{y}_5 = 1 + 12.7X$$

& we say that individual 5 deviates from the population intercept by -1.

Eg: Same fixed effects (fitted) model:

$$\hat{y} = 2 + 12.7X$$

Now suppose we had fitted a random intercepts & slope model & individual 5's BLUPS were $(-1, 2)$.

Then individual 5's fitted regression model would be:

$$\hat{y}_5 = 1 + 14.7X.$$

Generally there will be a BLUP for each group for each random effect in the model.

Variance - Covariance Matrices:

A var-covar matrix describes the variances & covariances between numerous variables. In the current context, we need to understand var-covar matrices so we can describe the structure of r.e.s in LME models with more than one r.e. per group.

If we have p variables, the var-covar matrix will be a $p \times p$, square, symmetric & positive definite matrix (pos-def means determinant > 0) of the form:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{12} & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & \\ \sigma_{1p} & \cdots & & \sigma_p^2 \end{bmatrix}$$

The main diagonal contains the variances, & the other elements are the covar between the ith & jth variables.

In other words, the $(i,j)^{th}$ element of Σ is the covar between variable i & variable j.

<u>Note</u>: 1. Symmetric $\rightarrow \sigma_{ij} = \sigma_{ji}$

2. $\sigma_{ii} =$ var $\forall\ i = 1, \ldots, p$.

Why does this matter to us? Recall that if we fit a random intercepts & slope LME, we have:

$$y_{ij} = \beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p + u_{0i} + u_{1i} X + \varepsilon_{ij}$$

where $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$

and

$(u_{0i}, u_{1i}) \sim N(\underset{\sim}{0}, \Sigma)$,

Where Σ is a var-covar matrix of the form:

$$\Sigma = \begin{bmatrix} \sigma_{u_0}^2 & \sigma_{01} \\ \sigma_{01} & \sigma_{u_1}^2 \end{bmatrix}$$

$\sigma^2_{u_0}$ is the var of u_{0i}

$\sigma^2_{u_1}$ is the var of u_{1i}

σ_{01} is the covar between u_0 & u_1.

<u>Models for R.E. Covariance Matrix</u>:

Just as we can model relationships between Xs & y, we may want to model how random effects impact on one another. We can do this by specifying various Σ structures.

We will only concentrate on situations where there are 2 r.es per group (ie random intercepts & slopes models).

1. Random slope & intercept do not depend on each other:

$$\Sigma = \begin{bmatrix} \sigma_{u_0}^2 & 0 \\ 0 & \sigma_{u_1}^2 \end{bmatrix} \quad \underline{ie}: \ \sigma_{01} = 0$$

This structure is known as variance-components.

2. R slope & intercept depend on one another:

$$\Sigma = \begin{bmatrix} \sigma_{u_0}^2 & \sigma_{01} \\ \sigma_{01} & \sigma_{u_1}^2 \end{bmatrix}$$

We allow the model to estimate σ_{01} from the data. This is known as an unstructured var-covar.

In case 1. we are saying that we believe u_0 & u_1 do not affect each other.

In case 2, we are saying that they do affect each other. (Eg: a larger u_0 leads to a larger or smaller u_1, or vice-versa).

It is generally best to start by assuming case 2., then checking whether we can simplify to case 1. We will see how to do this later.

When we have more than 2 res per cluster, the structures we can place on Σ increase. To see examples of what kind of structures are available, see the SAS help for proc mixed.